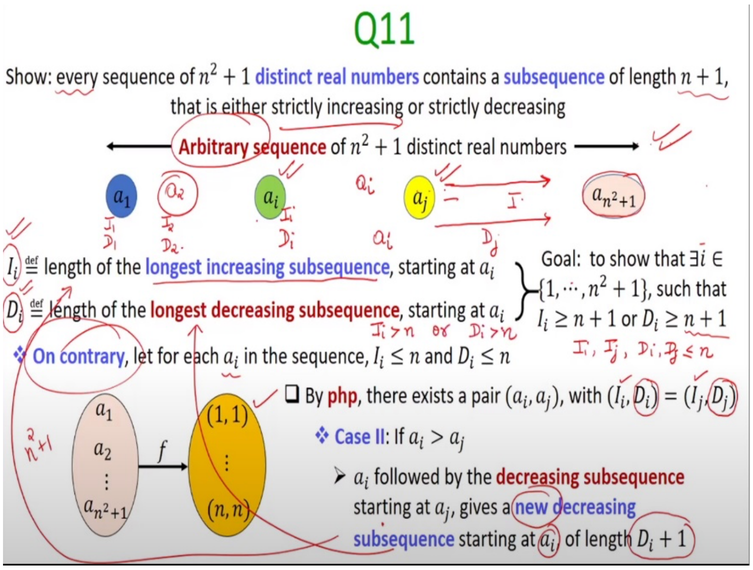

So now let us go to question number 11 which is really a very interesting question. Here we want to show the following that you take any sequence of $n^2 + 1$ distinct real numbers. They are arbitrary real numbers; may be positive, negative in any order you take them. The only condition is that they have to be distinct. Then the claim is that irrespective of the $n^2 + 1$ real numbers that you have in your sequence you always have a subsequence of length $n + 1$ which is either strictly increasing or strictly decreasing.

First of all what what is a strictly increasing sequence? A sequence of the form $(a_1, a_2, \ldots)$ where $a_1 < a_2 < a_3 < \cdots < a_{i-1} < a_i$ ... . Whereas if I have a sequence of the form $(a_1, a_2, a_3, \ldots)$ where $a_1 > a_2 > a_3 \ldots > a_{i-1} > a_i > \cdots$ then it is a strictly decreasing sequence.

Now what does a subsequence means? A subsequence mean here that the values may not be consecutive. That means I am allowed to miss few numbers. In the sense, say I take a sequence 1, 3, 0, -5, 2, 8 and so on. Then I can choose to pick 1 and then exclude 3 and 0 and -5. This is a subsequence. In the same way I can pick a subsequence saying 3, 2 and 8 that means I skip 0, I skip -5.

So what this question basically says is that irrespective of the way your $n^2 + 1$ distinct real numbers are chosen you always have a subsequence. By that I mean that you have a set of $n + 1$ values going from left to right but need not be in consecutive locations; some of the locations might be skipped. But the number of values are $n + 1$ such that if you view those $n + 1$ values they are either strictly increasing or strictly decreasing. That is what we have to prove.

Again, if you want to convince yourself whether this is indeed a true statement or not you can take some concrete values of n, try to draw any possible sequence of $n^2 + 1$ for that value of n and you can verify that this statement is true. But now we want to prove it for any arbitrary sequence. How do we do that? So let the arbitrary sequence of $n^2 + 1$ distinct real numbers be denoted by a_1 to a_{n^2+1}.

Why I am taking arbitrary here? Because I want to prove this statement for every sequence. So this is a universally quantified statement and to prove a universally quantified statement I can use the universal generalization principle by proving that a statement is true for some arbitrary element of the domain. My domain here is the set of all possible sequences of $n^2 + 1$ distinct real numbers. I am just taking one candidate element from that domain arbitrarily.

I do not know the exact values of a_1 ... a_j ... a_{n^2+1}. What I will do is to prove this statement, I will use pigeonhole principle along with proof by contradiction. So let me first define two values. I define I_i as the length of the longest increasing subsequence starting at a_i. So a_i will have some value depending upon what is the arbitrary sequence and it will have some various possible increasing subsequences starting at a_i.

One might be of length 1, a trivial increasing subsequence starting at a_i is the valuea_i itself. That is a subsequence of length 1. But I might be having a subsequence of say length 2 which is strictly increasing and starting at a_i. I might have a subsequence of length 3 starting at a_i and so on. So whatever is the length of the longest increasing subsequences starting at a_i, I am denoting by I_i.

In the same way, I define D_i as the length of the longest decreasing subsequence starting at a_i. I might have several strictly decreasing sequences starting at a_i. In fact the sequence a_i itself is a subsequence of length 1 which is strictly decreasing. But I might be having a subsequence of higher length which is strictly decreasing and starting at a_i. So the length of the longest decreasing subsequence starting at a_i I am denoting it as D_i. So that means with a_1, I have associated the values I_1 and D_1. With a_2, I would have associated the valueI_2 and D_2.

And similarly with a_i, I would have associated the value I_i and D_i, with a_j I would have associated the value I_j and D_j; and with a_{n^2+1} I would have associated the value I_{n^2+1} and D_{n^2+1}. It is easy to see that I_{n^2+1} will be 1, D_{n^2+1} is 1. Because I have only one sub sequence starting at a_{n^2+1} namely the value a_{n^2+1} itself.

It is both an increasing subsequence starting at a_{n^2+1} as well as it is a decreasing subsequence starting at a_{n^2+1} because there is nothing after the number a_{n^2+1}. Now, what is my goal? The question basically asks me to show that there always exist some i or some value a_i such that there either exists an increasing subsequence of length $n + 1$ that means I_i is greater than equal to $n + 1$ or there is a decreasing subsequence of length $n + 1$. That means D_i is $n + 1$.

I have to show the existence of one such number a_i in this subsequence. I prove that by assuming a contradiction. So assume that the statement is false and that means for each a_i in the sequence, the value $I_i$is at most n. That means you take any number in the sequence the maximum length increasing subsequence of length n and the maximum length decreasing subsequence is also of length n.

What does that mean? That means if I try to pair all I_i and D_i pairs then they can take the values in the range $(1,1)$ to (n, n) namely n^2 possible pairs. These are the possible values of (I_i, D_i) pairs. But how many numbers I have in the sequence? I have $n^2 + 1$ values in the sequence that I have chosen. That means I have more pigeons and less holes. What does that mean? So by PHP here I mean pigeonhole principle.

So pigeonhole principle guarantees me that you definitely have a pair of values here say a_i and a_j. Such that your I_i and I_j are same. That means the length of the longest increasing subsequence starting at a_i is the same as the length of the longest increasing subsequence starting at a_j. And in the same way the length of the longest decreasing subsequence starting at D_i is same as the length of the longest decreasing subsequence starting at D_j.

And as per my assumption I_i, I_j, D_i, D_j are all upper bounded by n because I assume the contradiction. Now how do I arrive at a contradiction here? So there could be two possible cases with respect to the magnitude of a_i and a_j.

The first case $a_i < a_j$. If that is the case, then what I can say is the following. I can say that you take the increasing subsequence starting at a_j; what is its length? Its length is I_j and if I put the value of a_i at the beginning of that subsequence then that gives me now a new increasing subsequence starting at a_i and of length I_{i+1}. But that goes against the assumption that the length of the longest subsequence starting at a_i wasI_i. So that is how I arrive at a contradiction.

On the other hand if I take the case when $a_i > a_j$ then I have to just give a symmetric argument. What I can say is the following. I know that there is a decreasing subsequence starting at a_j and its length is D_j. My claim is if you take that subsequence and put an a_i at the beginning then that now give me a new decreasing subsequence starting at a_i and the length of this new decreasing subsequence is $D\_\{i + 1\}$.

Which now goes against the assumption that the length of the longest decreasing subsequence starting at a_i was D_i. That means in both the cases I arrived at a contradiction and that shows that whatever I assumed here that means the value of each I_i and each D_i was upper bounded by n is incorrect. That means there is at least one $a_i$where either I_i is greater than n or D_i is greater than n. I do not know what exactly is that a_i. So I gave you a non-constructive proof here. But I argued that the existence of such a_i is guaranteed.

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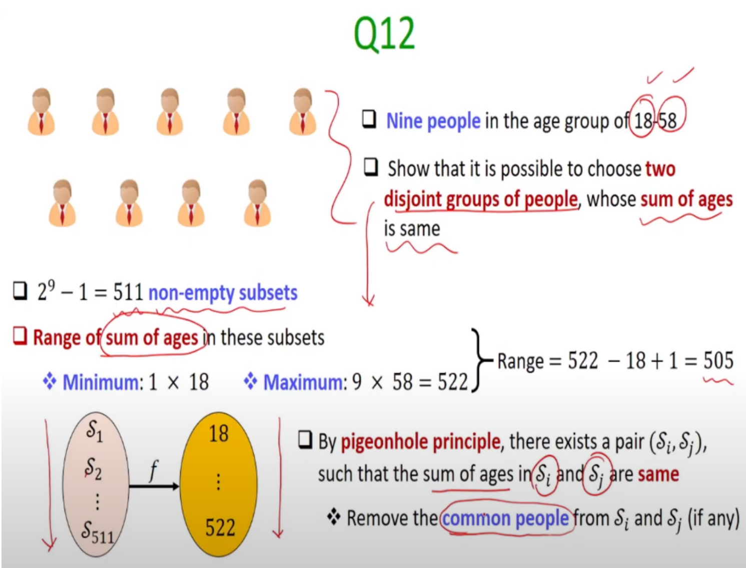

Now let us go to question 12. In this question we want to show the following you are arbitrarily picking 9 people in the age group of 18 to 58. That means the minimum age it is allowed is 18 the maximum age is allowed 58. Now we want to prove that irrespective of what exactly are their ages, as long as they are in the range 18 to 58 it is always possible to choose 2 disjoint groups of people out of this 9 people whose sum of ages is the same.

Again, we will do this by pigeonhole principle. So the first thing is since we want to argue about a non-empty set of people because when I want to consider the age of the people there have to be people in the group. So I have to focus on non-empty subset. So if I have 9 people then the number of non-empty groups that I can form out of those 9 people need not be disjoint is 511. And now what I can say about the range of the sum of ages in these 511 subsets.

If I consider the minimum sum of ages possible in a group it could be 18. This is possible only when I have a group of just consisting of one person and that person has age 18. That is a minimum possible sum. Whereas the maximum possible sum can occur when in my group I have all the 9 people person 1, person 2 and up to person 9 and each of them has age 58.

That is a maximum possible value of sum of the ages in a group we picked from 9 people. That means the range of possible sums here is 505. So now let us apply the pigeonhole principle. My pigeons are the various possible non-empty set of people that I can form out of this 9 group of 9 people. So I have 511 possible subsets and my holes are the range of sum of ages. That means what can be the sum of ages if I consider the various possible subsets given that the ages could be in the range 18 to 58.

So I have more pigeons than holes so by pigeonhole principle I can say that they always exists a pair of group S_i and S_j such that the sum of ages of the people in S_i and S_j are the same. But my question wants me to show that the group should be disjoint. So how do I argue that? I can always form disjoint groups of people out of this S_i and S_j. Well if they are already disjoint then I have showed the existence of 2 groups having the same sum of ages.

But if the sets S_i and S_j are not same; if they have some common people just remove the common people from both the set S_i and as well as S_j. The common people in the set S_i and S_j were contributing the same amount to the sum of ages in the set S_i as well as in the set S_j. So if I remove those common people the same amount will be removed from the sum of ages in S_i and S_j. And now where I will get 2 disjoint groups of people having the same sum of ages.

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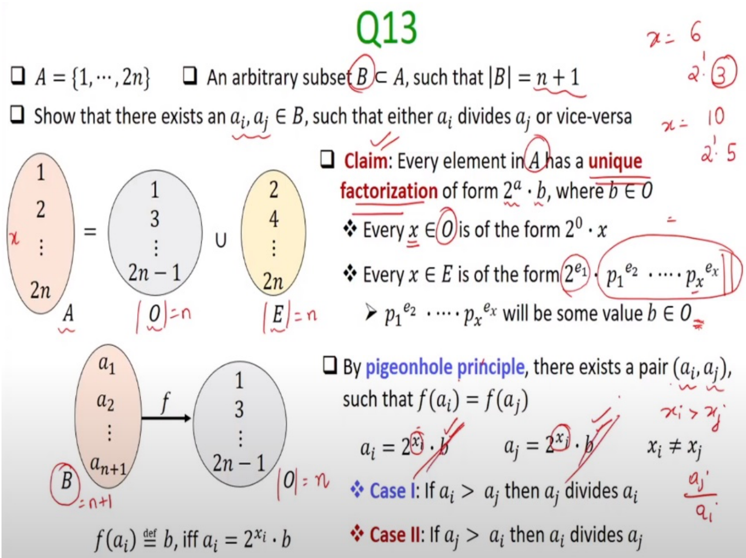

In question 13 you are given the set of A consisting of the numbers 1 to $2n$ and we want to show that if I pick an arbitrary subset B consisting of $n + 1$ elements from the set A and irrespective of the subset there always exist a pair of values such that one divides the other. And this is again a very interesting question. So for applying the pigeonhole principle what I do is I divide this set A

into 2 disjoint subsets namely the subset consisting of the odd values and the subset consisting of the even values.

Both of them will have the cardinality n. Now my claim is the following: you take any value in the set A, it has a unique factorization of the form that you have some power of 2 multiplied by the remaining value where the remaining value will be a number, specifically an odd number in the subset 1 to $2n - 1$. For example, if your number x that you are taking in the set A is already an odd number then I can write it in the form $2^0 * x$.

So in this case my a will be 0 and my x will be x itself. So my statement is true. Whereas if your x would have been say 6 then 6 can be written as 2 into 3. So you have 2^1 times an odd value. If your x is say 10 then you can write it as 2 $1 * 5$. If your x is say 20, then you can write it as $2^2 * 5$. So you can see that irrespective of the case, whether your x is odd or even, this claim is always true.

So for x being odd this statement is always true. But the statement is true even for a general x which is even because for such x where x is either 2, 4, or $2n$; I can express it in the form 2 power sum positive exponent e_1 followed by the remaining values. And this is because of the fundamental theorem of arithmetic that every integer has a unique prime factorization. The claim is that if I consider the remaining prime factorization here then that will be an odd value.

And that odd value will be in the set O here and it is easy to verify that. So that means this claim is true. Now based on this claim I have to apply the pigeonhole principle. For applying the pigeonhole principle I do the following. Let B be the arbitrary set of $n + 1$ values that I have chosen. And I mapped those arbitrary chosen values to the leftover value in its unique factorization that this claimed guarantees.

So a_1 will be written in the form of some $2^{x_1} * b$. So a_1 will be mapped to this b_1; a_2 will be written in the form of some 2^{x_2} into leftover thing. That left over thing is an odd number in the set O. So a_2 will be mapped to b_2 and so on. That is a mapping f here. Now what is the cardinality of set O? That is n; that means my number of holes is n. But the number of pigeons is $n + 1$. That means by pigeonhole principle it is guaranteed that there exists a pair of values a_i and a_j out of this n+1 values.

Where a_i is sum 2^{x_i} into some left over thing which is an odd value. And a_j is some 2^{x_j} multiplied by the same leftover value. I do not know the exact value of that left over odd value b. But that left over odd value b will be the same; that is a guarantee. And exponents $x_i \neq x_j$ because I am considering the distinct vales a_i and a_j. But what is guaranteed is that the leftover odd value here that is there as per this unique factorization claim will be the same. Now I have 2 possible cases if a_i is greater than a_j then clearly a_j divides a_i because if I divide a_j by a_i then the effect of b goes out and the exponent x_i is greater than exponent x_j. So, whatever is leftover that will be the quotient and the remainder will be 0.

Whereas if a_j is greater than a_i then again the effect of b vanishes and $2^{x_j}/2^{x_i}$ that will give you 0 remember. So irrespective of the case my statement is correct.

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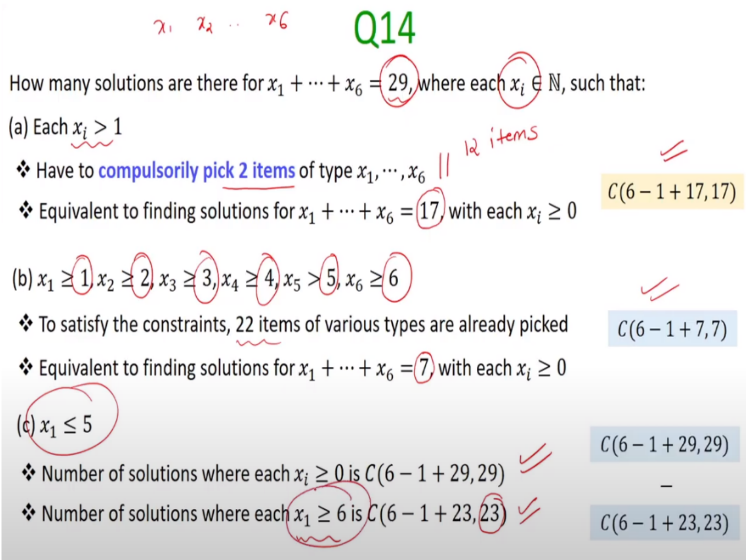

Now let us go the last question. Here we want to find out how many solutions are there for the equation $x_1 + \cdots x_6 = 29$ where there are various possible restrictions on x_i. So in part a, we have the restriction that each x_i has to be greater than 1. So you can imagine that you are given here bills of type x_1, bills of type x_2 and bills of type x_6. We have to pick total 29 bills with the restriction that you have to definitely pick more than one bill of each type.

That is the interpretation of this first restriction. That means I have to compulsorily pick 2 items of type $x_1, x_2 \dots x_6$. That means I had already picked 12 items compulsorily. That means now I am left over with the problem of picking 17 bills in total, out of this 6 different bill types where there are no restrictions. And remember as per the formula for the number of combinations with repetitions the answer is $C(6-1+17,17) = C(22,17)$.

In part b, the restriction is $x_i \geq i$. Again, if I interpret this restriction that means I have to definitely include one bill of type x_1, 2 bills of type x_2, 3 bills of type x_3, 4 pills of type x_4, 5 bills of type x_5 and 6 bills of x_6. That means I have already picked 22 bills of various types. That means now my goal was to pick 29 bills; 22 definitely I have already picked. So, I am left over with the problem of picking 7 bills where those 7 bills can be of type x_1, x_2 to x_6 in any possible order, no restrictions. So again, from the formula for number of r-combinations with repetition the answer will be $C(6-1+7,7) = C(12,7)$.

In part c, the restriction is that x_1 \le 5. So, what we do here is the following. We first find out the number of solutions where there is no restriction. That means x_1 maybe 0 as well; those solutions are also included in this quantity. And now I try to find out those solutions where this condition namely x_1 less than equal to 5 is violated. That means find the number of solutions where x_1 is greater than equal to 6.

That means definitely I have to pick 6 bills of type x_1 which further implies that now I am interested to pick the remaining 23 bills without putting any restriction that how many bills of different types I have to choose. The number of solutions for this case will be this. But this is not what we want. We want to find out the number of solutions which do not have this condition. So what I do? I subtract this value from the set of or from the number of solutions that I have without any restrictions and that will give me the answer.

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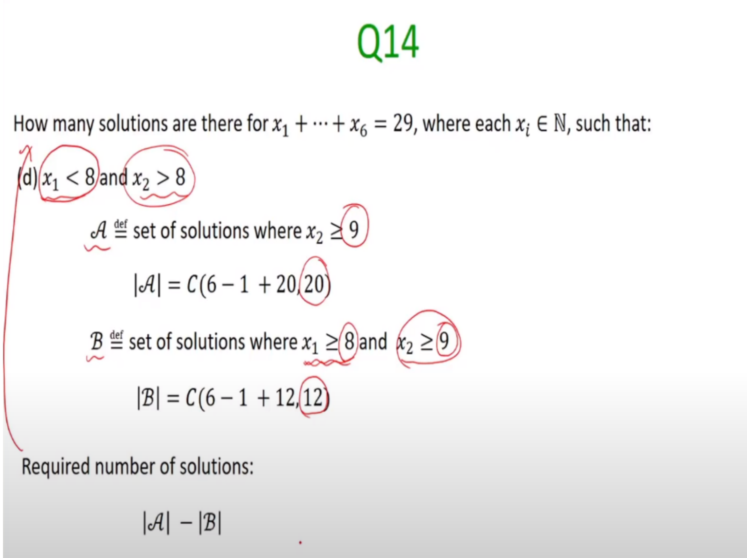

The last part here, my restrictions are $x\_1 \ < \ 8$ and $x\_2 \ > \ 8$. Let us first try to find out the number of solutions where $x\_2 \backslash ge$ 9. That means just try to satisfy the second restriction here. The number of solutions will be this because if x_2 is greater than equal to 9 that means 9 bills of type x_2 definitely have to be chosen. That means now I am left with the problem of picking 20 bills from bills of 6 types without any restrictions.

And now let us try to find out the number of solutions where this first restriction is violated, namely x_1 is greater than equal to 8 and x_2 is greater than equal to 9. So, what basically I am trying to do is the set A that I have defined here it has all those solutions where x_1 is less than 8 as well as x_1 is greater than 8. So, I am trying to take out those solutions where x_1 is greater than equal to 8 from this set A. I am denoting that set as B and the cardinality of the set B is this because if I am supposed to satisfy x_1 greater than equal to 8 and x_2 greater than equal to 9 that means I have already picked 17 bills. My goal will be now to pick 12 more bills from bills of 6 types without any restrictions. This will be the number of ways the number of solutions. And as I have said from the interpretation of the set A and B the required number of solutions is the difference of these 2 cardinalities which we can easily find out. So with that we finish our tutorial number 6. Thank you.